Product-to-sum identities

Sometimes you'll have the product of trig functions and you'll need to break up the product into a sum or difference, instead. You can use the product-to-sum identities to break the products into sums.

$$\sin\theta\cos\alpha = \frac{1}{2}\left[\sin(\theta+\alpha)+\sin(\theta-\alpha)\right]$$

$$\cos\theta\sin\alpha = \frac{1}{2}\left[\sin(\theta+\alpha)-\sin(\theta-\alpha)\right]$$

$$\cos\theta\cos\alpha = \frac{1}{2}\left[\cos(\theta+\alpha)+\cos(\theta-\alpha)\right]$$

$$\sin\theta\sin\alpha = \frac{1}{2}\left[\cos(\theta-\alpha)-\cos(\theta+\alpha)\right]$$

These identities are derived from the sum and difference identities for sine and cosine. We'll prove the first product-to-sum identity

$$\sin\theta\cos\alpha = \frac{1}{2}\left[\sin(\theta+\alpha)+\sin(\theta-\alpha)\right]$$

by starting with the expression on the right-hand side of the identity and deriving the expression on the left-hand side from it. The sum and difference identities for sine are

$$\sin(\theta+\alpha) = \sin\theta\cos\alpha+\cos\theta\sin\alpha$$

$$\sin(\theta-\alpha) = \sin\theta\cos\alpha-\cos\theta\sin\alpha$$

Therefore,

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$$\frac{1}{2}\left[\sin(\theta+\alpha)+\sin(\theta-\alpha)\right]=\frac{1}{2}\left[(\sin\theta\cos\alpha+\cos\theta\sin\alpha)+(\sin\theta\cos\alpha-\cos\theta\sin\alpha)\right]$$

$$\frac{1}{2}\left[\sin(\theta+\alpha)+\sin(\theta-\alpha)\right]=\frac{1}{2}\left[(\sin\theta\cos\alpha+\sin\theta\cos\alpha)+(\cos\theta\sin\alpha-\cos\theta\sin\alpha)\right]$$

$$\frac{1}{2}\left[\sin(\theta+\alpha)+\sin(\theta-\alpha)\right]=\frac{1}{2}(2\sin\theta\cos\alpha+0)$$

$$\frac{1}{2}\left[\sin(\theta+\alpha)+\sin(\theta-\alpha)\right]=\frac{1}{2}(2\sin\theta\cos\alpha)$$

$$\frac{1}{2}\left[\sin(\theta+\alpha)+\sin(\theta-\alpha)\right]=\sin\theta\cos\alpha$$

And this is the first product-to-sum identity. Now we'll show some applications of these identities.

Example

Express $\sin(7\theta)\sin(11\theta)$ in the form $a[\cos(b\theta)-\cos(c\theta)]$ where a, b and c are positive numbers.

By one of the product-to-sum identities,

$$\sin(7\theta)\sin(11\theta)=\frac{1}{2}\left[\cos(7\theta-11\theta)-\cos(7\theta+11\theta)\right]$$

$$\sin(7\theta)\sin(11\theta)=\frac{1}{2}\left[\cos(-4\theta)-\cos(18\theta)\right]$$

By the even identity for cosine,

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$$\cos(-4\theta) = \cos(4\theta)$$

Therefore,

$$\sin(7\theta)\sin(11\theta) = \frac{1}{2}\left[\cos(4\theta) - \cos(18\theta)\right]$$

This is in the prescribed form, with $a = 1/2$, $b = 4$, and $c = 18$.

Let's look a different kind of example.

Example

Find the exact values of $(\sin \pi/8)(\cos \pi/8)$, $\sin^2(\pi/8)$, and $\cos^2(\pi/8)$.

To compute $(\sin \pi/8)(\cos \pi/8)$, we can use the product-to-sum identity

$$\sin\theta\cos\alpha = \frac{1}{2}\left[\sin(\theta+\alpha) + \sin(\theta-\alpha)\right]$$

Setting both θ and α to $\pi/8$, we have

$$\left(\sin\frac{\pi}{8}\right)\left(\cos\frac{\pi}{8}\right) = \frac{1}{2}\left[\sin\left(\frac{\pi}{8}+\frac{\pi}{8}\right) + \sin\left(\frac{\pi}{8}-\frac{\pi}{8}\right)\right]$$

$$\left(\sin\frac{\pi}{8}\right)\left(\cos\frac{\pi}{8}\right) = \frac{1}{2}\left(\sin\frac{\pi}{4} + \sin 0\right)$$

We know that $\sin(\pi/4) = 1/\sqrt{2}$ and $\sin 0 = 0$, so

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$$\left(\sin\frac{\pi}{8}\right)\left(\cos\frac{\pi}{8}\right)=\frac{1}{2}\left(\frac{1}{\sqrt{2}}+0\right)=\frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)=\frac{1}{2\sqrt{2}}$$

To find $\sin^2(\pi/8)$, we can use the identity

$$\sin\theta\sin\alpha=\frac{1}{2}\left[\cos(\theta-\alpha)-\cos(\theta+\alpha)\right]$$

Setting both θ and α to $\pi/8$, this gives

$$\left(\sin\frac{\pi}{8}\right)\left(\sin\frac{\pi}{8}\right)=\frac{1}{2}\left[\cos\left(\frac{\pi}{8}-\frac{\pi}{8}\right)-\cos\left(\frac{\pi}{8}+\frac{\pi}{8}\right)\right]$$

$$\sin^2\left(\frac{\pi}{8}\right)=\frac{1}{2}\left(\cos 0-\cos\frac{\pi}{4}\right)$$

Since $\cos(\pi/4)=1/\sqrt{2}$ and $\cos 0=1$, we get

$$\sin^2\left(\frac{\pi}{8}\right)=\frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)=\frac{1}{2}\left(\frac{\sqrt{2}(1)-1(1)}{\sqrt{2}}\right)=\frac{1}{2}\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)=\frac{\sqrt{2}-1}{2\sqrt{2}}$$

To get the value of $\cos^2(\pi/8)$, we could use the identity

$$\cos\theta\cos\alpha=\frac{1}{2}\left[\cos(\theta+\alpha)+\cos(\theta-\alpha)\right]$$

However, since we've already found the value of $\sin^2(\pi/8)$, we could also get there by applying the basic Pythagorean identity to $\pi/8$:

$$\sin^2\left(\frac{\pi}{8}\right)+\cos^2\left(\frac{\pi}{8}\right)=1$$

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$$\cos^2\left(\frac{\pi}{8}\right) = 1 - \sin^2\left(\frac{\pi}{8}\right)$$

Substituting the value of $\sin^2(\pi/8)$ gives

$$\cos^2\left(\frac{\pi}{8}\right) = 1 - \left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right) = \frac{2\sqrt{2}(1) - 1\left(\sqrt{2}-1\right)}{2\sqrt{2}} = \frac{2\sqrt{2} - \sqrt{2} + 1}{2\sqrt{2}} = \frac{\sqrt{2}+1}{2\sqrt{2}}$$

Let's do another example.

Example

Find the exact value of $\cos(17\pi/12)\sin(\pi/12)$.

We can use the product-to-sum identity

$$\cos\theta \sin\alpha = \frac{1}{2}\left[\sin(\theta+\alpha) - \sin(\theta-\alpha)\right]$$

with $\theta = 17\pi/12$ and $\alpha = \pi/12$.

$$\left(\cos\frac{17\pi}{12}\right)\left(\sin\frac{\pi}{12}\right) = \frac{1}{2}\left[\sin\left(\frac{17\pi}{12}+\frac{\pi}{12}\right) - \sin\left(\frac{17\pi}{12}-\frac{\pi}{12}\right)\right]$$

$$\left(\cos\frac{17\pi}{12}\right)\left(\sin\frac{\pi}{12}\right) = \frac{1}{2}\left[\sin\left(\frac{18\pi}{12}\right) - \sin\left(\frac{16\pi}{12}\right)\right]$$

Reducing the fractions to lowest terms:

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$$\left(\cos\frac{17\pi}{12}\right)\left(\sin\frac{\pi}{12}\right)=\frac{1}{2}\left[\sin\left(\frac{3\pi}{2}\right)-\sin\left(\frac{4\pi}{3}\right)\right]$$

We know that $\sin(3\pi/2)=-1$. Also, an angle of $4\pi/3$ radians is in the third quadrant, so its reference angle is

$$\frac{4\pi}{3}-\pi=\frac{\pi}{3}$$

By the reference angle theorem (and using the fact that the sine of an angle in the third quadrant is negative), we get

$$\sin\frac{4\pi}{3}=-\sin\frac{\pi}{3}=-\frac{\sqrt{3}}{2}$$

Therefore,

$$\left(\cos\frac{17\pi}{12}\right)\left(\sin\frac{\pi}{12}\right)=\frac{1}{2}\left[-1-\left(-\frac{\sqrt{3}}{2}\right)\right]$$

$$\left(\cos\frac{17\pi}{12}\right)\left(\sin\frac{\pi}{12}\right)=\frac{1}{2}\left(-1+\frac{\sqrt{3}}{2}\right)$$

$$\left(\cos\frac{17\pi}{12}\right)\left(\sin\frac{\pi}{12}\right)=\frac{1}{2}\left[\frac{2(-1)+1\sqrt{3}}{2}\right]$$

$$\left(\cos\frac{17\pi}{12}\right)\left(\sin\frac{\pi}{12}\right)=\frac{-2+\sqrt{3}}{4}$$

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